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多孔介质中相互作用的 Brinkman-Forchheimer 流与 Darcy 流的空间衰减估计

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摘 要:研究了多孔介质中一类相互作用的 Brinkman-Forchheimer 流与 Darcy 流的空间衰减估计.首先定义一个加权的能量表达式,然后借助一些有用的引理推导出该能量表达式所满足的微分不等式,最后得到解的空间衰减估计结果.

关键词:多孔介质;Brinkman-Forchheimer 流;Darcy 流;空间衰减估计

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多孔介质在我们生活中是广泛存在的,多孔介质中流体方程组解的性态研究已经成为数学与力学领域的热点问题.目前已有的研究主要是集中在 Brinkman,Darcy 和 Forchheimer 方程组的模型上.在 NIELD 和 BEJAN[1]、STRAUGHAN[2]的书中广泛地讨论了多孔介质中的这些模型.文献[3－4]对多孔介质中流体方程组的结构稳定性进行了研究.近年来,文献[5－12]取得了一些新的关于结构稳定性的结果.以上文献只在有界区域内研究结构稳定性,当在无界区内研究空间衰减估计时,压力项的处理将会比较困难.

文献[13－14]主要研究无界区域内方程组的空间衰减估计,但他们所研究的方程组均是一个方程组,而在现实中,同一个区域中可能存在两个或多个相互作用的方程组,由于这些方程组之间还有相互作用,导致研究难度加大,AMES 等[15]研究了在合适的界面条件下相互作用的多孔介质方程组与黏性流体方程组解的加权能量函数的指数衰减估计,得到了空间衰减估计的结果.除文献[15]外基本无文献涉及.本文将讨论在一个空间中两种相互作用的流体方程组的空间衰减估计.

下面介绍本文所要考虑的问题:设 $\Omega=\Omega_1\cup\Omega_2$ 表示半无限柱体的内部,母线平行于 x_3 轴,x_1,x_3 平面以上的部分用 Ω_1 表示,x_1,x_3 平面以下的部分用 Ω_2 表示,L 表示 Ω_1 和 Ω_2 交界面.在平面 $x_3=0$ 上,Ω_1 是有界的,其侧面用 Γ_1 表示.同样的,在平面 $x_3=0$ 上,Ω_2 是有界的,其侧面用 Γ_2 表示.Ω 如图 1 所示.

定义 $\Omega_1=\{(x_1,x_2)\in D_1,x_3>0\}$,$\Omega_2=\{(x_1,x_2)\in D_2,x_3>0\}$,其中 D_1 和 D_2 分别表示 Ω_1 和 Ω_2 的横截面.由此可知在 Ω_1 上,$x_2>0$,在 Ω_2 上,$x_2<0$.假设流体在 Ω_1 部分由依赖于温度的 Brinkman-Forchheimer 流体方程控制,在 Ω_2 部分由依赖于温度的 Darcy 流体方程控制.

本文中,逗号表示求偏导数,对 x_k 的偏导数记为,k,$u_{,k}$ 表示为 $\frac{\partial u}{\partial x_k}$.重复拉丁字母下标表示从 1 到 3 求和,重复希腊字母下标表示从 1 到 2 求和.例如:$u_{i,i}=\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}$,$u_{\alpha,\alpha}=\sum_{\alpha=1}^{2}\frac{\partial u_\alpha}{\partial x_\alpha}$.

设 (u_i,T,P) 和 (v_i,S,q) 分别表示 $\Omega_1\times\{t>0\}$ 和 $\Omega_2\times\{t>0\}$ 上的流速、温度和压强.在 $\Omega_1\times\{t>0\}$ 上的 Brinkman-Forchheimer 方程组是(见文献[16])

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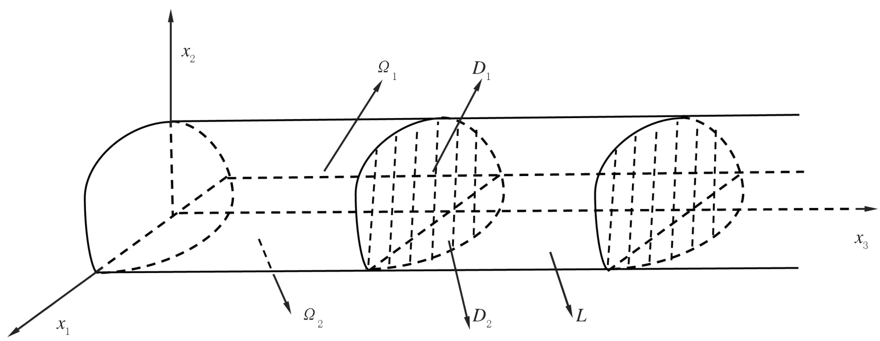

图1 半无限柱体Ω示意图

Fig.1 Diagrammatic sketch of semi-infinite cylinder Ω

$$\begin{cases} b\mid u\mid u_i-\mu\Delta u_i=-p_{,i}+g_iT, \\ \dfrac{\partial u_i}{\partial x_i}=0, \\ \dfrac{\partial T}{\partial t}+u_iT_{,i}=k\Delta T, \end{cases} \tag{1}$$

其中b是Forchheimer系数,μ是动力黏度系数,k是热扩散系数,g_i是重力向量,Δ为拉普拉斯算子.不失一般性,假设$|g|\leq 1$.

在$\Omega_2\times\{t>0\}$上的Darcy方程组是(见文献[17])

$$\begin{cases} \dfrac{\mu}{k}v_i=-q_{,i}+g_iS, \\ \dfrac{\partial v_i}{\partial x_i}=0, \\ \dfrac{\partial S}{\partial t}+v_iS_{,i}=k_S\Delta S, \end{cases} \tag{2}$$

其中k_S是渗透率.

边界条件和初始条件如下:

$$u_i=0,T=0,(x,t)\in\Gamma_1\times\{t>0\},v_in_i=0,S=0,(x,t)\in\Gamma_2\times\{t>0\},T=0,(x,t)\in\Omega_1\times\{t=0\},$$
$$S=0,(x,t)\in\Omega_2\times\{t=0\},u_3=f_3,T=g,(x,t)\in D_1\times\{x_3=0\}\times\{t>0\},$$
$$v_3=h_3,S=\tau,(x,t)\in D_2\times\{x_3=0\}\times\{t>0\}. \tag{3}$$

当x_3趋于无穷大时,假设下面的量关于x_1,x_2,t一致收敛于0,

$$\mid u\mid,\mid v\mid,\mid T\mid,\mid S\mid=O(1),$$
$$\mid u_3\mid,\mid v_3\mid,\mid\nabla u\mid,\mid\nabla T\mid,\mid\nabla S\mid,\mid p\mid,\mid q\mid=o(x_3^{-1}). \tag{4}$$

进一步,假设u_3在初始时刻沿着截面的流量为0,即$\int_{D_1}f_3\mathrm{d}A=0$.

最后,假设在交界面L上以下条件成立

$$u_2=v_2=I(x_1,x_2,t),T=S,k\frac{\partial T}{\partial x_2}=k_S\frac{\partial S}{\partial x_2},q=p-\mu\frac{\partial u_2}{\partial x_2},\frac{\partial u_i}{\partial x_2}=\frac{\alpha_1}{\sqrt{k}}u_i,i=1,3, \tag{5}$$

其中α_1是大于零的常数.

为了讨论方便,记$\Omega_i(z)=\Omega_i\times\{x_3>z\},i=1,2,D_i(z)=\Omega_i\times\{x_3=z\},i=1,2,L(z)=L\times\{x_3>z\},\Gamma_i(z)=\Gamma_i\times\{x_3>z\},i=1,2$.

相比文献[16]的研究,增加了非线性项$b|u|u_i$,由于非线性项的处理更复杂,此时文献[16]的二阶微分不等式的方法就不再适用,必须采用新的办法来解决这一问题,同时本文仅对初始值f_3加条件而没有对h_3

加任何限制,这些都是本文的特色.

1　加权能量表达式

在方程(1)$_1$ 两边同时乘以 u_i,并积分,可得

$$\int_0^t\int_{\Omega_1(z)}(\xi-z)u_i(b\mid u\mid u_i-\mu\Delta u_i+p_{,i}-g_iT)\mathrm{d}x\mathrm{d}\eta=0.\tag{6}$$

对于(6)式,由分部积分,可得

$$b\int_0^t\int_{\Omega_1(z)}(\xi-z)(u_iu_i)^{\frac{3}{2}}\mathrm{d}x\mathrm{d}\eta+\mu\int_0^t\int_{\Omega_1(z)}(\xi-z)u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\eta+\mu\int_0^t\int_{\Omega_1(z)}u_iu_{i,3}\mathrm{d}x\mathrm{d}\eta+\mu\int_0^t\int_{L(z)}(\xi-z)u_{i,2}u_i\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_{\Omega_1(z)}u_3p\mathrm{d}x\mathrm{d}\eta-\int_0^t\int_{L(z)}(\xi-z)u_2p\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_{\Omega_1(z)}(\xi-z)g_iu_iT\mathrm{d}x\mathrm{d}\eta=0.\tag{7}$$

由条件(5),可知

$$\mu\int_0^t\int_{L(z)}(\xi-z)u_{i,2}u_i\mathrm{d}A\mathrm{d}\eta=\frac{\alpha_1\mu}{\sqrt{k}}\int_0^t\int_{L(z)}(\xi-z)(u_1^2+u_3^2)\mathrm{d}A\mathrm{d}\eta+\mu\int_0^t\int_{L(z)}(\xi-z)u_{2,2}u_2\mathrm{d}A\mathrm{d}\eta.\tag{8}$$

由于

$$\int_0^t\int_{L(z)}(\xi-z)u_2p\mathrm{d}A\mathrm{d}\eta-\mu\int_0^t\int_{L(z)}(\xi-z)u_{2,2}u_2\mathrm{d}A\mathrm{d}\eta=\int_0^t\int_{L(z)}(\xi-z)qv_2\mathrm{d}A\mathrm{d}\eta=\int_0^t\int_{\Omega_2(z)}[(\xi-z)qv_i]_{,i}\mathrm{d}x\mathrm{d}\eta=-\frac{\mu}{k}\int_0^t\int_{\Omega_2(z)}(\xi-z)v_iv_i\mathrm{d}x\mathrm{d}\eta+\int_0^t\int_{\Omega_2(z)}qv_3\mathrm{d}x\mathrm{d}\eta+\int_0^t\int_{\Omega_2(z)}(\xi-z)g_iv_iS\mathrm{d}x\mathrm{d}\eta.\tag{9}$$

联合(7)～(9)式,可得

$$\Phi_1(z,t)=b\int_0^t\int_{\Omega_1(z)}(\xi-z)(u_iu_i)^{\frac{3}{2}}\mathrm{d}x\mathrm{d}\eta+\mu\int_0^t\int_{\Omega_1(z)}(\xi-z)u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\eta+\frac{\mu}{k}\int_0^t\int_{\Omega_2(z)}(\xi-z)v_iv_i\mathrm{d}x\mathrm{d}\eta+\frac{\alpha_i\mu}{k}\int_0^t\int_{L(z)}(\xi-z)(u_1^2+u_2^2)\mathrm{d}A\mathrm{d}\eta=-\mu\int_0^t\int_{\Omega_1(z)}u_iu_{i,3}\mathrm{d}x\mathrm{d}\eta+\int_0^t\int_{\Omega_1(z)}(\xi-z)g_iu_iT\mathrm{d}x\mathrm{d}\eta+\int_0^t\int_{\Omega_2(z)}(\xi-z)g_iv_iS\mathrm{d}x\mathrm{d}\eta+\int_0^t\int_{\Omega_1(z)}u_3p\mathrm{d}x\mathrm{d}\eta+\int_0^t\int_{\Omega_2(z)}v_3q\mathrm{d}x\mathrm{d}\eta.\tag{10}$$

下面,定义一个新的函数 $\varphi_1(z,t)=\int_0^t\int_{\Omega_1(z)}(\xi-z)T_{,i}T_{,i}\mathrm{d}x\mathrm{d}\eta+\int_0^t\int_{\Omega_2(z)}(\xi-z)S_{,i}S_{,i}\mathrm{d}x\mathrm{d}\eta$, 通过分部积分,可得

$$\varphi_1(z,t)=-\frac{1}{2k}\int_{\Omega_1(z)}(\xi-z)T^2\mathrm{d}x\mid_{\eta=t}+\frac{1}{2k}\int_0^t\int_{\Omega_1(z)}T^2u_3\mathrm{d}x\mathrm{d}\eta-\frac{1}{2k}\int_0^t\int_{L(z)}(\xi-z)T^2u_2\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_{\Omega_1(z)}TT_{,3}\mathrm{d}x\mathrm{d}\eta-\int_0^t\int_{L(z)}(\xi-z)TT_{,2}\mathrm{d}A\mathrm{d}\eta-\frac{1}{2k_S}\int_{\Omega_2(z)}(\xi-z)S^2\mathrm{d}x\mid_{\eta=t}+\frac{1}{2k_S}\int_0^t\int_{\Omega_2(z)}S^2v_3\mathrm{d}x\mathrm{d}\eta+\frac{1}{2k_S}\int_0^t\int_{L(z)}(\xi-z)S^2v_2\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_{\Omega_2(z)}SS_{,3}\mathrm{d}x\mathrm{d}\eta+\int_0^t\int_{L(z)}(\xi-z)SS_{,2}\mathrm{d}A\mathrm{d}\eta.\tag{11}$$

令

$$\Phi_2(z,t)=\int_0^t\int_{\Omega_1(z)}(\xi-z)T_{,i}T_{,i}\mathrm{d}x\mathrm{d}\eta+\int_0^t\int_{\Omega_2(z)}(\xi-z)S_{,i}S_{,i}\mathrm{d}x\mathrm{d}\eta+\frac{1}{2k}\int_{\Omega_1(z)}(\xi-z)T^2\mathrm{d}x\mid_{\eta=t}+\frac{1}{2k_S}\int_{\Omega_2(z)}(\xi-z)S^2\mathrm{d}x\mid_{\eta=t}.\tag{12}$$

则

$$\Phi_2(z,t)=\frac{1}{2k}\int_0^t\int_{\Omega_1(z)}T^2u_3\mathrm{d}x\mathrm{d}\eta-\int_0^t\int_{\Omega_1(z)}TT_{,3}\mathrm{d}x\mathrm{d}\eta+\frac{1}{2k_S}\int_0^t\int_{\Omega_2(z)}S^2v_3\mathrm{d}x\mathrm{d}\eta-\int_0^t\int_{\Omega_2(z)}SS_{,3}\mathrm{d}x\mathrm{d}\eta.\tag{13}$$

本文中,定义如下能量表达式:$\Phi(z,t)=\Phi_1(z,t)+\tilde{k}\Phi_2(z,t)$, 其中 $\tilde{k}$ 是稍后定义大于零的常数.

因此,有

$$\Phi(z,t)=b\int_0^t\int_{\Omega_1(z)}(\xi-z)(u_iu_i)^{\frac{3}{2}}\mathrm{d}x\mathrm{d}\eta+\mu\int_0^t\int_{\Omega_1(z)}(\xi-z)u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\eta+\frac{\mu}{k}\int_0^t\int_{\Omega_2(z)}(\xi-z)v_iv_i\mathrm{d}x\mathrm{d}\eta+$$
$$\frac{\alpha_1\mu}{\sqrt{k}}\int_0^t\int_{L(z)}(\xi-z)(u_1^2+u_2^2)\mathrm{d}A\mathrm{d}\eta+\tilde{k}\int_0^t\int_{\Omega_1(z)}(\xi-z)T_{,i}T_{,i}\mathrm{d}x\mathrm{d}\eta+\tilde{k}\int_0^t\int_{\Omega_2(z)}(\xi-z)S_{,i}S_{,i}\mathrm{d}x\mathrm{d}\eta+$$
$$\frac{\tilde{k}}{2k}\int_{\Omega_1(z)}(\xi-z)T^2\mathrm{d}x\Big|_{\eta=t}+\frac{\tilde{k}}{2k_S}\int_{\Omega_2(z)}(\xi-z)S^2\mathrm{d}x\Big|_{\eta=t}=-\mu\int_0^t\int_{\Omega_1(z)}u_iu_{i,3}\mathrm{d}x\mathrm{d}\eta+$$
$$\int_0^t\int_{\Omega_1(z)}(\xi-z)g_iu_iT\mathrm{d}x\mathrm{d}\eta+\int_0^t\int_{\Omega_2(z)}(\xi-z)g_iv_iS\mathrm{d}x\mathrm{d}\eta+\int_0^t\int_{\Omega_1(z)}u_3p\mathrm{d}x\mathrm{d}\eta+$$
$$\int_0^t\int_{\Omega_2(z)}v_3q\mathrm{d}x\mathrm{d}\eta+\frac{\tilde{k}}{2k}\int_0^t\int_{\Omega_1(z)}T^2u_3\mathrm{d}x\mathrm{d}\eta-\tilde{k}\int_0^t\int_{\Omega_1(z)}TT_{,3}\mathrm{d}x\mathrm{d}\eta+$$
$$\frac{\tilde{k}}{2k_S}\int_0^t\int_{\Omega_2(z)}S^2v_3\mathrm{d}x\mathrm{d}\eta-\tilde{k}\int_0^t\int_{\Omega_2(z)}SS_{,3}\mathrm{d}x\mathrm{d}\eta=\sum_{i=1}^{9}J_i. \tag{14}$$

2 重要的引理

引理 1 对于温度 T 与 S,有以下估计

$$\sup\{|T|,|S|\}\leq T_M, \tag{15}$$

其中 $T_M=\max\{\sup_{D_1\times[0,\infty]}|g|,\sup_{D_2\times[0,\infty]}|\tau|\}$.

证明 当 $x_3>0$ 时,由最大值原理可知,温度 T 与 S 的最大值不可能在边界 L 上取得见文献[18].因此最大值必然出现在 $x_3=0$ 处.则 $\sup\{|T|,|S|\}\leq T_M$.

引理 2 对于速度 u_3 与 v_3,满足以下等式

$$\int_{D_1(z)}u_3\mathrm{d}A=0, \tag{16}$$

$$\int_{D_2(z)}v_3\mathrm{d}A=0. \tag{17}$$

证明 显然,有 $\int_{\Omega_1(z)}u_{i,i}\mathrm{d}x+\int_{\Omega_2(z)}v_{i,i}\mathrm{d}x=0$. 上式由分部积分,可得 $-\int_{D_1(z)}u_3\mathrm{d}A-\int_{L(z)}u_2\mathrm{d}A-\int_{D_2(z)}v_3\mathrm{d}A+\int_{L(z)}v_2\mathrm{d}A=0$. 故可知

$$-\int_{D_1(z)}u_3\mathrm{d}A-\int_{D_2(z)}v_3\mathrm{d}A=0. \tag{18}$$

由于 $\int_{\Omega_1(z)}u_{i,i}\mathrm{d}x=0$,所以 $\int_{D_1(z)}u_3\mathrm{d}A=-\int_{L(z)}u_2\mathrm{d}A$.

又由于 u_2 在 $L(z)$ 取值与 z 无关,所以 $\int_{D_1(z)}u_3\mathrm{d}A$ 取值与 z 也无关. 因此

$$\int_{D_1(z)}u_3\mathrm{d}A=\int_{D_1}u_3\mathrm{d}A=\int_{D_1}f_3\mathrm{d}A=0. \tag{19}$$

联合(18)式和(19)式,可得 $\int_{D_1(z)}u_3\mathrm{d}A=0$, $\int_{D_2(z)}v_3\mathrm{d}A=0$.

引理 3[15] 对于固定的 x_2,x_3,有以下估计

$$\int_\iota u_1^2\mathrm{d}x_1\leq\frac{d^2(x_2)}{\pi^2}\int_\iota u_{1,1}^2\mathrm{d}x_1\leq\frac{d_0^2(x_2)}{\pi^2}\int_\iota u_{1,1}^2\mathrm{d}x_1, \tag{20}$$

其中 ι 是 $x_2=$常数和 $D_1(x_3)$ 的交集,$d(x_2)$ 是这个交集的长度,d_0 是 $d(x_2)$ 的极大值.

引理 4[15] 对于固定的 x_1,x_3,有以下估计

$$\int_{\bar\iota}u_2^2\mathrm{d}x_2\leq\frac{4\rho^2(x_1)}{\pi^2}\int_{\bar\iota}u_{2,2}^2\mathrm{d}x_2\leq\frac{4\rho_0^2(x_1)}{\pi^2}\int_{\bar\iota}u_{2,2}^2\mathrm{d}x_2, \tag{21}$$

其中 $\bar{\iota}$ 是 $x_1=$ 常数和 $D_1(x_3)$ 的交集,$\rho(x_1)$ 是这个交集的长度,ρ_0 是 $\rho(x_1)$ 的极大值.

引理 5[15]　假设 $\tilde{\Omega}$ 为 $\mathbf{R}^3$ 的一个带有 Lipschitz 边界的有界区域,令 χ 作为 $\tilde{\Omega}$ 区域的一个有界函数,其均值为0,则存在一个依赖于 $\tilde{\Omega}$ 的向量函数 ω_i 和一个常数 C 满足如下式子:

$$\omega_{i,i}=\chi,x\in\tilde{\Omega},\omega_i=0,x\in\partial\tilde{\Omega},\tag{22}$$

$$\int_{\tilde{\Omega}}\omega_{i,j}\omega_{i,j}\mathrm{d}x\leq C\int_{\tilde{\Omega}}\chi^2\mathrm{d}x.\tag{23}$$

3　空间衰减估计

首先给出(14)式中的 $\sum_{i=1}^{9}J_i$ 一个估计.

运用引理3和引理4的结果,有

$$J_1=-\mu\int_0^t\int_{\Omega_1(z)}u_iu_{i,3}\mathrm{d}x\mathrm{d}\eta\leq\mu\left(\frac{\mathrm{d}_0^2}{\pi^2}\int_0^t\int_{\Omega_1(z)}u_{1,3}^2\mathrm{d}x\mathrm{d}\eta\cdot\int_0^t\int_{\Omega_1(z)}u_{1,1}^2\mathrm{d}x\mathrm{d}\eta\right)^{\frac{1}{2}}+\mu\left(\frac{4\rho_0^2}{\pi^2}\int_0^t\int_{\Omega_1(z)}u_{2,3}^2\mathrm{d}x\mathrm{d}\eta\cdot\int_0^t\int_{\Omega_1(z)}u_{2,2}^2\mathrm{d}x\mathrm{d}\eta\right)^{\frac{1}{2}}+\mu\left(\int_0^t\int_{\Omega_1(z)}u_{3,3}^2\mathrm{d}x\mathrm{d}\eta\cdot\int_0^t\int_{\Omega_1(z)}u_3^2\mathrm{d}x\mathrm{d}\eta\right)^{\frac{1}{2}}.$$

下面来证明 $\int_0^t\int_{\Omega_1(z)}u_3^2\mathrm{d}x\mathrm{d}\eta\leq\frac{d_0^2}{\pi^2}\int_0^t\int_{\Omega_1(z)}u_{3,1}^2\mathrm{d}x\mathrm{d}\eta$. $\int_0^t\int_{\Omega_1(z)}u_3^2\mathrm{d}x\mathrm{d}\eta=\int_0^t\int_z^{\infty}\int_{D_1(\xi)}u_3^2\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}\xi\mathrm{d}\eta$. 而 $\int_{D_1(\xi)}u_3^2\mathrm{d}x_1\mathrm{d}x_2=\int_{D_1(z)}u_3^2(x_1,x_2,\xi)\mathrm{d}x_1\mathrm{d}x_2$.

由于 D_1 是平面有界区域,运用(20)式,可知 $\int_{D_1}u_3^2(x_1,x_2,\xi)\mathrm{d}x_1\mathrm{d}x_2\leq\frac{d_0^2}{\pi^2}\int_{D_1}u_{3,1}^2(x_1,x_2,\xi)\mathrm{d}x_1\mathrm{d}x_2$.

故 $\int_0^t\int_{\Omega_1(z)}u_3^2\mathrm{d}x\mathrm{d}\eta\leq\frac{d_0^2}{\pi^2}\int_0^t\int_{\Omega_1(z)}u_{3,1}^2\mathrm{d}x\mathrm{d}\eta$.

由 $\int_0^t\int_{\Omega_1(z)}u_3^2\mathrm{d}x\mathrm{d}\eta\leq\frac{d_0^2}{\pi^2}\int_0^t\int_{\Omega_1(z)}u_{3,1}^2\mathrm{d}x\mathrm{d}\eta$ 和 Young 不等式,可得

$$J_1\leq k_1\int_0^t\int_{\Omega_1(z)}u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\eta,\tag{24}$$

其中 k_1 为可计算的大于零的常数.

下面确定 J_2 的界.

$$J_2=\int_0^t\int_{\Omega_1(z)}(\xi-z)g_iu_iT\mathrm{d}x\mathrm{d}\eta\leq\frac{\varepsilon_1}{2}\int_0^t\int_{\Omega_1(z)}(\xi-z)u_iu_i\mathrm{d}x\mathrm{d}\eta+\frac{1}{2\varepsilon_1}\int_0^t\int_{\Omega_1(z)}(\xi-z)T^2\mathrm{d}x\mathrm{d}\eta.\tag{25}$$

运用引理3和引理4的结果,有

$$\int_0^t\int_{\Omega_1(z)}(\xi-z)u_iu_i\mathrm{d}x\mathrm{d}\eta\leq\frac{d_0^2}{\pi^2}\int_0^t\int_{\Omega_1(z)}(\xi-z)(u_{1,1}^2+u_{3,1}^2)\mathrm{d}x\mathrm{d}\eta+\frac{4\rho_0^2}{\pi^2}\int_0^t\int_{\Omega_1(z)}(\xi-z)u_{2,2}^2\mathrm{d}x\mathrm{d}\eta.\tag{26}$$

联合(25)、(26)式,可得

$$J_2\leq\frac{\varepsilon_1(d_0^2+4\rho_0^2)}{2\pi^2}\int_0^t\int_{\Omega_1(z)}(\xi-z)u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\eta+\frac{1}{\lambda^2\varepsilon_1}\int_0^t\int_{\Omega_1(z)}(\xi-z)T_{,i}T_{,i}\mathrm{d}x\mathrm{d}\eta,\tag{27}$$

其中 ε_1 为大于零的常数,λ 为下面问题的最小特征值 $U_{,\alpha\alpha}+\lambda U=0,x\in D_1,U=0,x\in\partial D_1\cap\Gamma_1,U_{,\alpha}=0,x\in\partial D_1\cap\Gamma_1$.

由 Schwarz 不等式,可得

$$J_3=\int_0^t\int_{\Omega_2(z)}(\xi-z)g_iv_iS\mathrm{d}x\mathrm{d}\eta\leq\frac{\varepsilon_2}{2}\int_0^t\int_{\Omega_2(z)}(\xi-z)v_iv_i\mathrm{d}x\mathrm{d}\eta+\frac{1}{2\varepsilon_2\nu}\int_0^t\int_{\Omega_2(z)}(\xi-z)S_{,i}S_{,i}\mathrm{d}x\mathrm{d}\eta,\tag{28}$$

其中 ε_2 为大于零的常数,ν 为下面问题的最小特征值 $V_{,\alpha\alpha}+\nu V=0,x\in D_2,V=0,x\in\partial D_2\cap\Gamma_2,V_{,\alpha}=0,$

$x\in\partial D_2\cap\Gamma_2$.

运用引理1的结果,有

$$J_6+J_8=\frac{\tilde{k}}{2k}\int_0^t\int_{\Omega_1(z)}T^2u_3\,\mathrm{d}x\,\mathrm{d}\eta+\frac{\tilde{k}}{2k_S}\int_0^t\int_{\Omega_2(z)}S^2v_3\,\mathrm{d}x\,\mathrm{d}\eta\le\frac{\tilde{k}^2T_M^2}{4k^2}\int_0^t\int_{\Omega_1(z)}T^2\,\mathrm{d}x\,\mathrm{d}\eta+\frac{1}{4}\int_0^t\int_{\Omega_1(z)}u_3^2\,\mathrm{d}x\,\mathrm{d}\eta+$$
$$\frac{\tilde{k}^2T_M^2}{4k_S^2}\int_0^t\int_{\Omega_2(z)}S^2\,\mathrm{d}x\,\mathrm{d}\eta+\frac{1}{4}\int_0^t\int_{\Omega_2(z)}v_3^2\,\mathrm{d}x\,\mathrm{d}\eta\le\frac{\tilde{k}^2T_M^2}{4\lambda k^2}\int_0^t\int_{\Omega_1(z)}T_{,i}T_{,i}\,\mathrm{d}x\,\mathrm{d}\eta+$$
$$\frac{1}{4}\frac{d_0^2}{\pi^2}\int_0^t\int_{\Omega_1(z)}u_{3,1}^2\,\mathrm{d}x\,\mathrm{d}\eta+\frac{\tilde{k}^2T_M^2}{4\nu k_S^2}\int_0^t\int_{\Omega_2(z)}S_{,i}S_{,i}\,\mathrm{d}x\,\mathrm{d}\eta+\frac{1}{4}\int_0^t\int_{\Omega_2(z)}v_3^2\,\mathrm{d}x\,\mathrm{d}\eta.\tag{29}$$

同理,可得

$$J_7+J_9=-\tilde{k}\int_0^t\int_{\Omega_1(z)}TT_{,3}\,\mathrm{d}x\,\mathrm{d}\eta-\tilde{k}\int_0^t\int_{\Omega_2(z)}SS_{,3}\,\mathrm{d}x\,\mathrm{d}\eta\le$$
$$\frac{\tilde{k}}{\sqrt{\lambda}}\int_0^t\int_{\Omega_1(z)}T_{,i}T_{,i}\,\mathrm{d}x\,\mathrm{d}\eta+\frac{\tilde{k}}{\sqrt{\nu}}\int_0^t\int_{\Omega_2(z)}S_{,i}S_{,i}\,\mathrm{d}x\,\mathrm{d}\eta.\tag{30}$$

联合(24)～(30)式,可得

$$J_1+J_2+J_3+J_6+J_7+J_8+J_9\le\frac{\varepsilon_1(d_0^2+4\rho_0^2)}{2\pi^2}\int_0^t\int_{\Omega_1(z)}(\xi-z)u_{i,j}u_{i,j}\,\mathrm{d}x\,\mathrm{d}\eta+$$
$$\frac{1}{2\varepsilon_1\lambda}\int_0^t\int_{\Omega_1(z)}(\xi-z)T_{,i}T_{,i}\,\mathrm{d}x\,\mathrm{d}\eta\le\frac{\varepsilon_2}{2}\int_0^t\int_{\Omega_1(z)}(\xi-z)v_iv_i\,\mathrm{d}x\,\mathrm{d}\eta+$$
$$\frac{1}{2\varepsilon_2\nu}\int_0^t\int_{\Omega_2(z)}S_{,i}S_{,i}\,\mathrm{d}x\,\mathrm{d}\eta+k_2\left[-\frac{\partial\Phi}{\partial z}\right],\tag{31}$$

其中 k_2 为可计算的大于零的常数.

下面给出 J_4 的界.考虑

$$\int_0^t\int_z^{z+a}\int_{D_1(\delta)}pu_3\,\mathrm{d}x\,\mathrm{d}\eta=\int_0^t\int_z^{z+a}\int_{D_1(\delta)}p\omega_{j,j}\,\mathrm{d}x\,\mathrm{d}\eta,\tag{32}$$

其中 $\omega_{j,j}=u_3,x\in D_1(z)\times\{z<x_3<z+a\}$;$\omega_j=0,x\in\partial D_1(z)\times\{z<x_3<z+a\}$ 或 $\{x_3=z\}$ 或 $\{x_3=z+a\}$.

对于(32)式,由分部积分,可得

$$\int_0^t\int_z^{z+a}\int_{D_1(\delta)}u_3p\,\mathrm{d}x\,\mathrm{d}\eta=-\int_0^t\int_z^{z+a}\int_{D_1(\delta)}\omega_jp_{,j}\,\mathrm{d}x\,\mathrm{d}\eta=\int_0^t\int_z^{z+a}\int_{D_1(\delta)}\omega_j(b|u|u_i-\mu u_{i,jj}-g_iT)\,\mathrm{d}x\,\mathrm{d}\eta\le$$
$$b\left[\int_0^t\int_z^{z+a}\int_{D_1(\delta)}(\omega_j\omega_j)^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}\eta\right]^{\frac{1}{3}}\cdot\left[\int_0^t\int_z^{z+a}\int_{D_1(\delta)}(u_ju_j)^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}\eta\right]^{\frac{2}{3}}+\mu\left[\int_0^t\int_z^{z+a}\int_{D_1(\delta)}(\omega_{j,j})^2\,\mathrm{d}x\,\mathrm{d}\eta\right]^{\frac{1}{2}}\cdot$$
$$\left(\int_0^t\int_z^{z+a}\int_{D_1(\delta)}u_{i,j}u_{i,j}\,\mathrm{d}x\,\mathrm{d}\eta\right)^{\frac{1}{2}}+\left[\int_0^t\int_z^{z+a}\int_{D_1(\delta)}\omega_j\omega_j\,\mathrm{d}x\,\mathrm{d}\eta\right]^{\frac{1}{2}}\cdot\left(\int_0^t\int_z^{z+a}\int_{D_1(\delta)}T^2\,\mathrm{d}x\,\mathrm{d}\eta\right)^{\frac{1}{2}}.\tag{33}$$

又因为

$$\int_0^t\int_z^{z+a}\int_{D_1(\delta)}(\omega_j\omega_j)^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}\eta\le\left(\int_0^t\int_z^{z+a}\int_{D_1(\delta)}(\omega_j\omega_j)\,\mathrm{d}x\,\mathrm{d}\eta\right)^{\frac{1}{2}}\cdot\left(\int_0^t\int_z^{z+a}\int_{D_1(\delta)}\omega_j\omega_j\,\mathrm{d}x\,\mathrm{d}\eta\right)^{\frac{1}{2}}\le$$
$$\hat{k}^{\frac{1}{2}}\left(\int_0^t\int_z^{z+a}\int_{D_1(\delta)}\omega_{j,i}\omega_{j,i}\,\mathrm{d}x\,\mathrm{d}\eta\right)^{\frac{1}{4}}\cdot\left(\int_0^t\int_z^{z+a}\int_{D_1(\delta)}\omega_j\omega_j\,\mathrm{d}x\,\mathrm{d}\eta\right)^{\frac{5}{4}}.$$

其中 $\hat{k}$ 为常数.且满足如下 Poincaré 不等式 $\int_\Omega(\omega_j\omega_j)^2\,\mathrm{d}x\le\hat{k}(\int_\Omega\omega_{j,i}\omega_{j,i}\,\mathrm{d}x)^{\frac{1}{2}}(\int_\Omega\omega_j\omega_j\,\mathrm{d}x)^{\frac{3}{2}}$.所以

$$\int_z^{z+a}\int_{D_1(\delta)}(\omega_j\omega_j)^{\frac{3}{2}}\,\mathrm{d}x\le C^{\frac{1}{2}}\hat{k}^{\frac{1}{2}}\left(\int_z^{z+a}\int_{D_1(\delta)}u_3^2\,\mathrm{d}x\right)^{\frac{1}{4}}\cdot\Lambda^{-\frac{5}{4}}\left(\int_z^{z+a}\int_{D_1(\delta)}\omega_{j,i}\omega_{j,i}\,\mathrm{d}x\right)^{\frac{5}{4}}\le$$
$$\hat{k}^{\frac{1}{2}}\Lambda^{-\frac{5}{4}}\left(\int_z^{z+a}\int_{D_1(\delta)}u_3^2\,\mathrm{d}x\right)^{\frac{3}{2}}\le\hat{k}^{\frac{1}{2}}\Lambda^{-\frac{5}{4}}|a|^{\frac{1}{3}}|D|^{\frac{1}{3}}\int_{D_1(z)}(u_iu_i)^{\frac{3}{2}}\,\mathrm{d}x.\tag{34}$$

其中 Λ 是如下问题的第一特征值 $\Delta\varphi+\Lambda\varphi=0, x\in D_1(Z)\cap\{z<x_3<z+a\}, \varphi=0, x\in\Gamma_1(z)\cap\{z<x_3\leq z+a\}, \varphi=0, x\in D_1(z)\cap\{x_3=z, x_3=z+a\}$.

联合(33)、(34)式，可得

$$J_4\leq k_3\left[-\frac{\partial\Phi(z,t)}{\partial z}\right], \tag{35}$$

其中 k_3 为可计算的大于零的常数.

下面给出 J_5 的界.由引理 2 可知，$\int_{D_2(z)} v_3\,\mathrm{d}x=0$. 运用引理 5 的结论，有

$$\int_0^t\int_z^{z+a}\int_{D_2(\delta)} qv_3\,\mathrm{d}x\,\mathrm{d}\eta=\int_0^t\int_z^{z+a}\int_{D_2(\delta)} q\tilde{\omega}_{j,j}\,\mathrm{d}x\,\mathrm{d}\eta. \tag{36}$$

其中 $\tilde{\omega}_{j,j}=v_3, x\in D_2(z)\times\{z<x_3<z+a\}, \tilde{\omega}_j=0, x\in\partial D_2(z)\times\{z<x_3<z+a\}$ 或 $\{x_3=z\}$ 或 $\{x_3=z+a\}$. 由分部积分，可得

$$\begin{aligned}\int_0^t\int_z^{z+a}\int_{D_2(\delta)} qv_3\,\mathrm{d}x\,\mathrm{d}\eta&=-\int_0^t\int_z^{z+a}\int_{D_2(\delta)}\tilde{\omega}_j q_{,j}\,\mathrm{d}x\,\mathrm{d}\eta=\int_0^t\int_z^{z+a}\int_{D_2(\delta)}\tilde{\omega}_j\left(\frac{\mu}{k}v_j-g_iS\right)\mathrm{d}x\,\mathrm{d}\eta\leq\\
&\frac{\mu}{k}\left(\int_0^t\int_z^{z+a}\int_{D_2(\delta)}\tilde{\omega}_j\tilde{\omega}_j\,\mathrm{d}x\,\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_z^{z+a}\int_{D_2(\delta)}v_jv_j\,\mathrm{d}x\,\mathrm{d}\eta\right)^{\frac{1}{2}}+\\
&\left(\int_0^t\int_z^{z+a}\int_{D_2(\delta)}\tilde{\omega}_j\tilde{\omega}_j\,\mathrm{d}x\,\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_z^{z+a}\int_{D_2(\delta)}S^2\,\mathrm{d}x\,\mathrm{d}\eta\right)^{\frac{1}{2}}\leq\\
&\frac{\mu C^{\frac{1}{2}}}{k\tilde{\Lambda}^{\frac{1}{2}}}\left(\int_0^t\int_z^{z+a}\int_{D_2(\delta)}v_3^2\,\mathrm{d}x\,\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_z^{z+a}\int_{D_2(\delta)}v_jv_j\,\mathrm{d}x\,\mathrm{d}\eta\right)^{\frac{1}{2}}+\\
&\frac{C^{\frac{1}{2}}}{\tilde{\Lambda}^{\frac{1}{2}}}\left(\int_0^t\int_z^{z+a}\int_{D_2(\delta)}v_3^2\,\mathrm{d}x\,\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_z^{z+a}\int_{D_2(\delta)}S_{,j}S_{,j}\,\mathrm{d}x\,\mathrm{d}\eta\right)^{\frac{1}{2}},\end{aligned} \tag{37}$$

其中 $\tilde{\Lambda}$ 是如下问题的第一特征值 $\Delta\varphi+\tilde{\Lambda}\varphi=0, x\in D_2(z)\cap\{z<x_3<z+a\}, \varphi=0, x\in\Gamma_2(z)\cap\{z<x_3\leq z+a\}, \varphi=0, x\in D_2(z)\cap\{x_3=z, x_3=z+a\}$.

由(37)式，可得

$$J_5\leq k_4\left[-\frac{\partial\Phi(z,t)}{\partial z}\right], \tag{38}$$

其中 k_4 为可计算的大于零的常数.

联合(14)、(31)、(35)和(38)式，并取 $\varepsilon_1=\frac{\pi^2\mu}{d_0^2+4\rho_0^2}, \varepsilon_2=\frac{\mu}{k}, \tilde{k}=\max\left\{\frac{d_0^2+4\rho_0^2}{\pi^2\mu\lambda}, \frac{k}{\mu\iota}\right\}$，可得

$$\Phi(z,t)\leq k_5\left[-\frac{\partial\Phi(z,t)}{\partial z}\right], \tag{39}$$

其中 k_5 为可计算的大于零的常数.

(39)式积分得

$$\Phi(z,t)\leq\Phi(0,t)\exp(-k_5z). \tag{40}$$

不等式(40)显示，当 $z\to\infty$ 时，$\Phi(z,t)$ 衰减且趋于零.

定理 1 (14)式中所定义的能量表达式 $\Phi(z,t)$ 满足如下空间衰减估计：$\Phi(z,t)\leq\Phi(0,t)\exp(-k_5z)$.

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Spatial decay estimates for the Brinkman-Forchheimer fluid interfacing with a Darcy fluid in porous medium

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Abstract: The Spatial decay estimates for the Brinkman-Forchheimer fluid interfacing with a Darcy fluid in porous medium is studied. Firstly, the weighted energy expression is defined. Then, with the help of some useful lemmas, the differential inequality satisfied by the energy expression is derived. Finally, the result of the spatial decay estimates of the solution is obtained.

Keywords: porous medium; Brinkman-Forchheimer fluid; Darcy fluid; spatial decay estimates

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