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具有时变系数的非局部混合抛物系统解的爆破

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摘　要:研究了非线性边界条件下具有时变系数和吸收项的非局部混合抛物系统解的爆破问题.运用微分不等式技巧,得到了高维空间上非线性边界条件下具有时变系数和吸收项的非局部混合抛物系统全局解的条件.同时,通过构造能量表达式,应用 Sobolev 不等式等技巧,推出了爆破发生时解的爆破时间下界的估计.

关键词:爆破;抛物系统;全局存在性;时变系数;吸收项

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抛物方程和抛物系统解的爆破研究近来取得了丰硕的成果[1-9].早期的研究成果主要集中在三维空间上齐次边界条件和 Robin 边界条件下考虑解的全局存在、爆破时间上下界的估计、爆破率、解的渐近性等.之后,考虑了在高维空间上非线性边界条件下解的性态研究.近年来,具有时变或空变系数的局部和非局部抛物方程和抛物系统解的爆破问题受到了学者们广泛关注,主要考虑在一定的边值条件下爆破解的全局存在性和上下界估计等.一定意义上,非局部的抛物方程和抛物系统比局部的情况更有实际价值,因而探讨非局部的抛物方程和抛物系统解的爆破有更强的理论和实际意义.同时因为局部的数学模型理论和方法不适用于非局部的情况,故对于非局部的研究存在不少困难.有关爆破解的界的估计,目前研究上界的方法较多,而下界较少.

文献[2]研究了一类非线性反应扩散方程解的爆破问题:$(g(u))_t=\nabla\cdot(\rho(|\nabla u|^2)\nabla u)+k(t)f(u)$,$(x,t)\in\Omega\times(0,t^*)$,在齐次 Dirichlet 边界条件下,作者得到了高维空间上解的上下界估计和解的全局存在条件.

文献[3]研究了非线性边界条件下多孔介质抛物方程解的爆破问题:$u_t=\Delta u^m+u^p\int_\Omega v^q\mathrm{d}x-u^s$,$(x,t)\in\Omega\times(0,t^*)$,在齐次 Dirichlet 和 Neumann 边界条件下,作者分别得到了三维空间上爆破发生时解的爆破时间的下界估计.

文献[4]研究了依赖于时间的抛物系统解的爆破问题:

$$\begin{cases}u_t=\Delta u^m+k_1(t)f_1(v),(x,t)\in\Omega\times(0,t^*),\\ v_t=\Delta v^n+k_2(t)f_2(u),(x,t)\in\Omega\times(0,t^*),\end{cases}$$

在齐次 Dirichlet 边界条件下,作者得到了三维空间上解的全局存在的条件.同时在某些约束条件下得到了三维空间上解的爆破时间的上界和下界估计.

文献[5]研究了如下抛物系统爆破问题:

$$\begin{cases}u_t=\sum_{i,j=1}^n(a^{ij}(x)u_{x_i})_{x_j}-f_1(u,v),(x,t)\in\Omega\times(0,t^*),\\ v_t=\sum_{i,j=1}^n(b^{ij}(x)v_{x_i})_{x_j}-f_2(u,v),(x,t)\in\Omega\times(0,t^*),\end{cases}$$

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在非线性边界条件下，得到了高维空间上解的爆破条件以及爆破发生时解的爆破时间的上界和下界估计.

受以上文献的启发，本文研究非线性边界条件下具有时变系数和吸收项的非局部混合抛物系统解的全局存在性和爆破问题：

$$\begin{cases} u_t = \Delta u^m + k_1(t) u^p \int_\Omega v^q \mathrm{d}x - u^\alpha, (x,t) \in \Omega \times (0,t^*), \\ v_t = (\rho(|\nabla v|^2) v_{,i})_{,i} + k_2(t) v^r \int_\Omega u^s \mathrm{d}s - v^\beta, (x,t) \in \Omega \times (0,t^*), \\ \dfrac{\partial u}{\partial n} = g_1(u), \dfrac{\partial v}{\partial n} = g_2(v), (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, x \in \Omega, \end{cases} \tag{1}$$

其中 $m,p,q,r,s>1,\alpha,\beta>0,\rho(|\nabla v|^2)\geq 1,g_1(u)\geq 0,g_2(v)\geq 0,k_i(t)>0(i=1,2)$，$\Omega$ 是高维空间 $\mathbf{R}^n(n\geq 3)$ 中的一个有界凸区域，Δ 代表拉普拉斯算子，∇ 代表梯度算子.$\partial\Omega$ 是区域 Ω 的边界，t^* 代表可能的爆破时间.$\dfrac{\partial u}{\partial n},\dfrac{\partial v}{\partial n}$ 分别是 u,v 在边界 $\partial\Omega$ 上的外法向量导数，假设其足够光滑.

抛物方程和抛物系统解的爆破研究在物理学、生物学、天文学、化学、航天等领域有着广泛的应用.目前尚未发现研究(1)的相关文献.本文的目标是研究其在高维空间上非线性边界条件下解的全局存在性条件以及爆破发生时解的爆破时间的下界估计.如何恰当处理高维空间、非局部项、吸收项以及非线性边界条件对解的爆破影响是本文的难点也是其亮点.

1　全局存在性

本文推导需要用到下面两个引理.

引理 1[6]　设 Ω 是 $\mathbf{R}^n(n\geq 3)$ 上的有界凸区域，则对于 $u\in C^1(\Omega),s>0$，有如下不等式：

$$\int_{\partial\Omega} u^s \mathrm{d}A \leq \frac{n}{\rho_0}\int_\Omega u^s \mathrm{d}x + \frac{sd}{\rho_0}\int_\Omega u^{s-1}|\nabla u|\mathrm{d}x, \tag{2}$$

其中 $\rho_0=\min\limits_{x\in\partial\Omega}(x,n),d=\max\limits_{x\in\Omega}|x|$.

引理 2[10]　Sobolev 不等式

$$\int_\Omega u^{\frac{(\sigma+m-1)n}{n-2}}\mathrm{d}x \leq C^{\frac{2n}{n-2}} 2^{\frac{n}{n-2}-1}\left[\left(\int_\Omega u^{\sigma+m-1}\mathrm{d}x\right)^{\frac{n}{n-2}} + \left(\int_\Omega |\nabla u^{\frac{\sigma+m-1}{2}}|^2\mathrm{d}x\right)^{\frac{n}{n-2}}\right], \tag{3}$$

$$\int_\Omega v^{\frac{\sigma n}{n-2}}\mathrm{d}x \leq C^{\frac{2n}{n-2}} 2^{\frac{n}{n-2}-1}\left[\left(\int_\Omega v^\sigma \mathrm{d}x\right)^{\frac{n}{n-2}} + \left(\int_\Omega |\nabla v^{\frac{\sigma}{2}}|^2\mathrm{d}x\right)^{\frac{n}{n-2}}\right], \tag{4}$$

其中 $C=C(n,\Omega)$ 是一个与 n 和 Ω 有关的 Sobolev 嵌入常数.

定理 1　假设满足如下条件：

$$0\leq g_1(\xi)\leq b_1\xi^{s_1}, 0\leq g_2(\xi)\leq b_2\frac{\xi^{s_2}}{\rho(|\nabla\xi|^2)}, \xi>0, b_i>0, \rho(|\nabla\xi|^2)\geq 1, s_i,m,p,q,r,s>1,$$
$$\alpha>\max\{p+q,r+s,m+2s_1-2\},\beta>\max\{p+q,r+s,2s_2-1\},$$
$$\frac{k_i'(t)}{k_i(t)}\leq -l_i, l_i>0(i=1,2), k_i(t)>0, t\geq 0. \tag{5}$$

则问题(1)的解在任何有限时间都是有界的，即问题(1)是全局存在的.

证明　首先定义辅助函数：

$$\varphi(t)=k_1(t)\int_\Omega u^2\mathrm{d}x + k_2(t)\int_\Omega v^2\mathrm{d}x, \tag{6}$$

运用散度定理，首先对(6)式求导数并注意到(5)式，得：

$$\varphi'(t)=k'_1(t)\int_\Omega u^2\mathrm{d}x + 2k_1(t)\int_\Omega uu_t\mathrm{d}x + k'_2(t)\int_\Omega v^2\mathrm{d}x + 2k_2(t)\int_\Omega vv_t\mathrm{d}x \leq -L\varphi(t)+$$
$$2k_1(t)\int_\Omega u\Delta u^m\mathrm{d}x + 2k_1^2(t)\int_\Omega u^{p+1}\mathrm{d}x\int_\Omega v^q\mathrm{d}x - 2k_1(t)\int_\Omega u^{1+\alpha}\mathrm{d}x +$$

$$2k_2(t)\int_\Omega v(\rho(|\nabla v|^2)v_{,i})_{,i}\mathrm{d}x+2k_2^2(t)\int_\Omega v^{r+1}\mathrm{d}x\int_\Omega u^s\mathrm{d}x-2k_2(t)\int_\Omega v^{1+\beta}\mathrm{d}x. \tag{7}$$

其中 $L=\min\{l_1,l_2\}$.

对于(7)式右边第 2 项,由散度定理和(2)式,有

$$\int_\Omega u\Delta u^m\mathrm{d}x=\int_{\partial\Omega}u\frac{\partial u^m}{\partial n}\mathrm{d}A-\int_\Omega\nabla u\cdot\nabla u^m\mathrm{d}x\leq mb_1\int_{\partial\Omega}u^{m+s_1}\mathrm{d}A-\frac{4m}{(m+1)^2}\int_\Omega|\nabla u^{\frac{m+1}{2}}|^2\mathrm{d}x\leq$$

$$\frac{nmb_1}{\rho_0}\int_\Omega u^{m+s_1}\mathrm{d}x+\frac{(m+s_1)dmb_1}{\rho_0}\int_\Omega u^{m+s_1-1}|\nabla u|\mathrm{d}x-\frac{4m}{(m+1)^2}\int_\Omega|\nabla u^{\frac{m+1}{2}}|^2\mathrm{d}x. \tag{8}$$

对于(8)式右边第 2 项,由 Hölder 不等式和 Young 不等式,得:

$$\int_\Omega u^{m+s_1-1}|\nabla u|\mathrm{d}x\leq\frac{1}{2\varepsilon_1}\int_\Omega u^{m+2s_1-1}\mathrm{d}x+\frac{2\varepsilon_1}{(m+1)^2}\int_\Omega|\nabla u^{\frac{m+1}{2}}|^2\mathrm{d}x, \tag{9}$$

其中 ε_1 是正数,定义见后面文字.

于是,由(8)式和(9)式,得到:

$$2k_1(t)\int_\Omega u\Delta u^m\mathrm{d}x\leq r_1k_1(t)\int_\Omega u^{m+s_1}\mathrm{d}x+r_2k_1(t)\int_\Omega u^{m+2s_1-1}\mathrm{d}x+r_3k_1(t)\int_\Omega|\nabla u^{\frac{m+1}{2}}|^2\mathrm{d}x, \tag{10}$$

其中:$r_1=\frac{2nmb_1}{\rho_0}$,$r_2=\frac{(m+s_1)dmb_1}{\rho_0}\frac{1}{\varepsilon_1}$,$r_3=\frac{(m+s_1)dmb_1}{\rho_0}\times\frac{4\varepsilon_1}{(m+1)^2}-\frac{8m}{(m+1)^2}$.

对于(7)式右边第 5 项,由散度定理和(2)式,可得:

$$\int_\Omega v(\rho(|\nabla v|^2)v_{,i})_{,i}\mathrm{d}x=\int_{\partial\Omega}v\rho(|\nabla v|^2)\frac{\partial v}{\partial n}\mathrm{d}A-\int_\Omega\rho(|\nabla v|^2)|\nabla v|^2\mathrm{d}x\leq$$

$$\frac{nb_2}{\rho_0}\int_\Omega v^{1+s_2}\mathrm{d}x+\frac{(1+s_2)db_2}{\rho_0}\int_\Omega v^{s_2}|\nabla v|\mathrm{d}x-\int_\Omega|\nabla v|^2\mathrm{d}x. \tag{11}$$

对于(11)式右边第 2 项,由 Hölder 不等式和 Young 不等式,得:

$$\int_\Omega v^{s_2}|\nabla v|\mathrm{d}x\leq\frac{1}{2\varepsilon_2}\int_\Omega v^{2s_2}\mathrm{d}x+\frac{\varepsilon_2}{2}\int_\Omega|\nabla v|^2\mathrm{d}x, \tag{12}$$

其中 ε_2 是正数,定义见后面文字.

于是由(11)、(12)式有

$$2k_2(t)\int_\Omega v(\rho(|\nabla v|^2)v_{,i})_{,i}\mathrm{d}x\leq\lambda_1k_2(t)\int_\Omega v^{1+s_2}\mathrm{d}x+\lambda_2k_2(t)\int_\Omega v^{2s_2}\mathrm{d}x+\lambda_3k_2(t)\int_\Omega|\nabla v|^2\mathrm{d}x, \tag{13}$$

其中:$\lambda_1=\frac{2nb_2}{\rho_0}$,$\lambda_2=\frac{(1+s_2)db_2}{\rho_0}\frac{1}{\varepsilon_2}$,$\lambda_3=\frac{(1+s_2)db_2}{\rho_0}\varepsilon_2-2$.

对于(7)式右边第 3 项,由 Hölder 不等式和 Young 不等式,有

$$2k_1^2(t)\int_\Omega u^{p+1}\mathrm{d}x\int_\Omega v^q\mathrm{d}x\leq2|\Omega|k_1^2(t)\left(\int_\Omega u^{p+q+1}\mathrm{d}x\right)^{\frac{p+1}{p+q+1}}\left(\int_\Omega v^{p+q+1}\mathrm{d}x\right)^{\frac{q}{p+q+1}}\leq$$

$$\tilde{k}(t)k_1(t)\int_\Omega u^{p+q+1}\mathrm{d}x+\tilde{k}(t)k_2(t)\int_\Omega v^{p+q+1}\mathrm{d}x, \tag{14}$$

其中 $\tilde{k}(t)=2|\Omega|k_1^{2-\frac{p-1}{p+q+1}}(t)k_2^{-\frac{q}{p+q+1}}(t)$.

同样地,对于(7)式右边第 6 项,由 Hölder 不等式和 Young 不等式,有

$$2k_2^2(t)\int_\Omega v^{r+1}\mathrm{d}x\int_\Omega u^s\mathrm{d}x\leq\tilde{\tilde{k}}(t)k_2(t)\int_\Omega v^{r+s+1}\mathrm{d}x+\tilde{\tilde{k}}(t)k_1(t)\int_\Omega u^{r+s+1}\mathrm{d}x, \tag{15}$$

其中 $\tilde{\tilde{k}}(t)=2|\Omega|k_2^{2-\frac{r+1}{r+s+1}}(t)k_1^{-\frac{s}{r+s+1}}(t)$.

联立(7)式、(10)式及(13)～(15)式,有

$$\varphi'(t)\leq-L\varphi(t)+r_1k_1(t)\int_\Omega u^{m+s_1}\mathrm{d}x+r_2k_1(t)\int_\Omega u^{m+2s_1-1}\mathrm{d}x+r_3k_1(t)\int_\Omega|\nabla u^{\frac{m+1}{2}}|^2\mathrm{d}x+$$

$$\tilde{k}(t)k_1(t)\int_\Omega u^{p+q+1}\mathrm{d}x+\tilde{k}(t)k_2(t)\int_\Omega v^{p+q+1}\mathrm{d}x-2k_1(t)\int_\Omega u^{1+\alpha}\mathrm{d}x+$$

$$\lambda_1 k_2(t)\int_\Omega v^{1+s_2}\mathrm{d}x+\lambda_2 k_2(t)\int_\Omega v^{2s_2}\mathrm{d}x+\lambda_3 k_2(t)\int_\Omega|\nabla v|^2\mathrm{d}x+$$
$$\tilde{\tilde{k}}(t)k_2(t)\int_\Omega v^{r+s+1}\mathrm{d}x+\tilde{\tilde{k}}(t)k_1(t)\int_\Omega u^{r+s+1}\mathrm{d}x-2k_2(t)\int_\Omega v^{1+\beta}\mathrm{d}x. \tag{16}$$

选取合适的 $\varepsilon_1,\varepsilon_2$ 使得 $r_3\leq 0,\lambda_3\leq 0$. 于是，(16)式化为：

$$\varphi'(t)\leq -L\varphi(t)+r_1k_1(t)\int_\Omega u^{m+s_1}\mathrm{d}x+r_2k_1(t)\int_\Omega u^{m+2s_1-1}\mathrm{d}x+\tilde{k}(t)k_1(t)\int_\Omega u^{p+q+1}\mathrm{d}x+$$
$$\tilde{\tilde{k}}(t)k_1(t)\int_\Omega u^{r+s+1}\mathrm{d}x-2k_1(t)\int_\Omega u^{1+\alpha}\mathrm{d}x+\lambda_1k_2(t)\int_\Omega v^{1+s_2}\mathrm{d}x+\lambda_2k_2(t)\int_\Omega v^{2s_2}\mathrm{d}x+$$
$$\tilde{k}(t)k_2(t)\int_\Omega v^{p+q+1}\mathrm{d}x+\tilde{\tilde{k}}(t)k_2(t)\int_\Omega v^{r+s+1}\mathrm{d}x-2k_2(t)\int_\Omega v^{1+\beta}\mathrm{d}x. \tag{17}$$

由 Hölder 不等式和 Young 不等式，得：

$$\int_\Omega u^{m+s_1}\mathrm{d}x\leq\frac{m+s_1-2}{m+2s_1-3}\int_\Omega u^{m+2s_1-1}\mathrm{d}x+\frac{s_1-1}{m+2s_1-3}\int_\Omega u^2\mathrm{d}x, \tag{18}$$

$$\int_\Omega u^{m+2s_1-1}\mathrm{d}x\leq\frac{m+2s_1-3}{\alpha-1}\varepsilon_3\int_\Omega u^{1+\alpha}\mathrm{d}x+\frac{\alpha-m-2s_1+2}{\alpha-1}\varepsilon_3^{-\frac{m+2s_1-3}{\alpha-m-2s_1+2}}\int_\Omega u^2\mathrm{d}x, \tag{19}$$

$$\int_\Omega u^{p+q+1}\mathrm{d}x\leq\frac{p+q-1}{\alpha-1}\varepsilon_4\int_\Omega u^{1+\alpha}\mathrm{d}x+\frac{\alpha-p-q}{\alpha-1}\varepsilon_4^{-\frac{p+q-1}{\alpha-p-q}}\int_\Omega u^2\mathrm{d}x, \tag{20}$$

$$\int_\Omega u^{r+s+1}\mathrm{d}x\leq\frac{r+s-1}{\alpha-1}\varepsilon_5\int_\Omega u^{1+\alpha}\mathrm{d}x+\frac{\alpha-r-s}{\alpha-1}\varepsilon_5^{-\frac{r+s-1}{\alpha-r-s}}\int_\Omega u^2\mathrm{d}x, \tag{21}$$

$$\int_\Omega v^{1+s_2}\mathrm{d}x\leq\frac{1}{2}\int_\Omega v^{2s_2}\mathrm{d}x+\frac{1}{2}\int_\Omega v^2\mathrm{d}x, \tag{22}$$

$$\int_\Omega v^{2s_2}\mathrm{d}x\leq\frac{2s_2-2}{\beta-1}\varepsilon_6\int_\Omega v^{1+\beta}\mathrm{d}x+\frac{\beta-2s_2+1}{\beta-1}\varepsilon_6^{-\frac{2s_2-2}{\beta-2s_2+1}}\int_\Omega v^2\mathrm{d}x, \tag{23}$$

$$\int_\Omega v^{p+q+1}\mathrm{d}x\leq\frac{p+q-1}{\beta-1}\varepsilon_7\int_\Omega v^{1+\beta}\mathrm{d}x+\frac{\beta-p-q}{\beta-1}\varepsilon_7^{-\frac{p+q-1}{\beta-p-q}}\int_\Omega v^2\mathrm{d}x, \tag{24}$$

$$\int_\Omega v^{r+s+1}\mathrm{d}x\leq\frac{r+s-1}{\beta-1}\varepsilon_8\int_\Omega v^{1+\beta}\mathrm{d}x+\frac{\beta-r-s}{\beta-1}\varepsilon_8^{-\frac{r+s-1}{\beta-r-s}}\int_\Omega v^2\mathrm{d}x, \tag{25}$$

其中 $\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6,\varepsilon_7,\varepsilon_8$ 均是正数，定义见后面文字.

联立(17)～(25)式，有

$$\varphi'(t)\leq(-L+r_4+\lambda_4)\varphi(t)-(2-r_5)k_1(t)\int_\Omega u^{1+\alpha}\mathrm{d}x-(2-\lambda_5)k_2(t)\int_\Omega v^{1+\beta}\mathrm{d}x, \tag{26}$$

其中：$r_4=\dfrac{\alpha-p-q}{\alpha-1}\varepsilon^{-\frac{p+q-1}{\alpha-p-q}}\tilde{k}(t)+\dfrac{\alpha-r-s}{\alpha-1}\varepsilon_5^{-\frac{r+s-1}{\alpha-r-s}}\tilde{\tilde{k}}(t)+\dfrac{s_1-1}{m+2s_1-3}r_1+\dfrac{\alpha-m-2s_1+2}{\alpha-1}\varepsilon_3^{-\frac{m+2s_1-3}{\alpha-m-2s_1+2}}\times\left(\dfrac{m+s_1-2}{m+2s_1-3}r_1+r_2\right)$,

$$\lambda_4=\frac{\beta-2s_2+1}{\beta-1}\varepsilon_6^{-\frac{2s_2-2}{\beta-2s_2+1}}\left(\frac{1}{2}\lambda_1+\lambda_2\right)+\frac{\beta-p-q}{\beta-1}\varepsilon_7^{-\frac{p+q-1}{\beta-p-q}}\tilde{k}(t)+\frac{\beta-r-s}{\beta-1}\varepsilon_8^{-\frac{r+s-1}{\beta-r-s}}\tilde{\tilde{k}}(t)+\frac{1}{2}\lambda_1,$$

$$r_5=\frac{m+2s_1-3}{\alpha-1}\varepsilon_3\left(\frac{m+s_1-2}{m+2s_1-3}r_1+r_2\right)+\frac{p+q-1}{\alpha-1}\varepsilon_4\tilde{k}(t)+\frac{r+s-1}{\alpha-1}\varepsilon_5\tilde{\tilde{k}}(t),$$

$$\lambda_5=\frac{2s_2-2}{\beta-1}\varepsilon_6\left(\frac{1}{2}\lambda_1+\lambda_2\right)+\frac{p+q-1}{\beta-1}\varepsilon_7\tilde{k}(t)+\frac{r+s-1}{\beta-1}\varepsilon_8\tilde{\tilde{k}}(t).$$

选取合适的 $\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6,\varepsilon_7,\varepsilon_8$，使得 $2-r_5>0,2-\lambda_5>0$.

于是，由 Hölder 不等式，得到：

$$\int_\Omega u^{1+\alpha}\mathrm{d}x\geq\left(\int_\Omega u^2\mathrm{d}x\right)^{\frac{1+\alpha}{2}}|\Omega|^{\frac{1-\alpha}{2}}, \tag{27}$$

$$\int_\Omega v^{1+\beta}\mathrm{d}x \geq \left(\int_\Omega v^2\mathrm{d}x\right)^{\frac{1+\beta}{2}} |\Omega|^{\frac{1-\beta}{2}}. \tag{28}$$

联立(26)～(28)式,得到:

$$\varphi'(t) \leq (-L+r_4+\lambda_4)\varphi(t) - CK_1\varphi(t)^{1+K_2} = \varphi(t)[-L+r_4+\lambda_4-CK_1\varphi(t)^{K_2}], \tag{29}$$

其中:$K_1=\min\{(2-r_5)(k_1(t))^{\frac{1-\alpha}{2}}|\Omega|^{\frac{1-\alpha}{2}},(2-\lambda_5)(k_2(t))^{\frac{1-\beta}{2}}|\Omega|^{\frac{1-\beta}{2}}\}$,$K_2=\min\{\frac{\alpha-1}{2},\frac{\beta-1}{2}\}$,$C$ 为正常数.

(29)式表明 u 在 $\varphi(t)$ 测度下对于任意的 $t(t>0)$ 都不会爆破,即全局存在.否则,如果在某个 t^* 爆破,即

$$\lim_{t\to(t^*)^-}\varphi(t)=+\infty.$$

由(29)式,易知 $\varphi'(t)\leq 0,\forall t\in[t_0,t^*)$.从而有,$\varphi(t)\leq\varphi(t_0)$.当 $t\to(t^*)^-$ 时取极限,得:

$$\lim_{t\to(t^*)}\varphi(t)=+\infty\leq\varphi(t_0).$$

矛盾.定理 1 得证.

2 爆破时间的下界

假设满足如下条件:

$$0\leq g_1(\xi)\leq b_1\xi^{s_1},0\leq g_2(\xi)\leq b_2\frac{\xi^{s_2}}{\rho(|\nabla\xi|^2)},\xi>0,b_i>0,\rho(|\nabla\xi|^2)>1,s_i,m,p,q,r,s>1,$$

$$\alpha>m+2s_1-2,m<\min\{p+q,r+s\},\beta>2s_2-1,\frac{k'_i(t)}{k_i(t)}\leq a_i,a_i>0(i=1,2),k_i(t)>0,t\geq 0. \tag{30}$$

构造如下辅助函数:

$$\phi(t)=k_1^\delta(t)\int_\Omega u^\sigma\mathrm{d}x+k_2^\delta(t)\int_\Omega v^\sigma\mathrm{d}x, \tag{31}$$

其中 $\sigma>\max\left\{\frac{n(r+s-m)}{2},\frac{n(r+s-1)}{2},\frac{n(p+q-m)}{2},\frac{n(p+q-1)}{2},1\right\}$.

有如下定理.

定理 2 假设 $u(x,t)$,$v(x,t)$ 是问题(1)、(30)式在有界凸区域的经典的非负解.则(31)中定义的能量满足微分不等式 $\phi'(t)\leq K_6\phi(t)+K_7(t)\phi(t)^{\xi_1}+K_8(t)\phi(t)^{\xi_2}+K_9(t)\phi(t)^{\xi_3}$,由此可得爆破时间 t^* 的下界为 $t^*\geq\Theta^{-1}(S)$,其中 $K_6,K_7(t),K_8(t),K_9(t),\xi_1,\xi_2,\xi_3,\Theta,S$ 均在后面定义,Θ^{-1} 是 Θ 的反函数.

证明 对(31)式求导数并注意到(30)式,得:

$$\begin{aligned}\phi'(t)=&\delta k_1^{\delta-1}(t)k'_1(t)\int_\Omega u^\sigma\mathrm{d}x+\sigma k_1^\delta(t)\int_\Omega u^{\sigma-1}u_t\mathrm{d}x+\delta k_2^{\delta-1}(t)k'_2(t)\int_\Omega v^\sigma\mathrm{d}x+\sigma k_2^\delta(t)\int_\Omega v^{\sigma-1}v_t\mathrm{d}x\leq\\&\delta a\phi(t)+\sigma k_1^\delta(t)\int_\Omega u^{\sigma-1}\Delta u^m\mathrm{d}x+\sigma k_1^{\delta+1}(t)\int_\Omega u^{\sigma+p-1}\mathrm{d}x\int_\Omega v^q\mathrm{d}x-\sigma k_1^\delta(t)\int_\Omega u^{\sigma+\alpha-1}\mathrm{d}x+\\&\sigma k_2^\delta(t)\int_\Omega v^{\sigma-1}(\rho(|\nabla v|^2)v_{,i})_{,i}\mathrm{d}x+\sigma k_2^{\delta+1}(t)\int_\Omega v^{\sigma+r-1}\mathrm{d}x\int_\Omega u^s\mathrm{d}x-\sigma k_2^\delta(t)\int_\Omega v^{\sigma+\beta-1}\mathrm{d}x.\end{aligned} \tag{32}$$

其中 $a=\max\{a_1,a_2\}$.

对于(32)式右边第 2 项,由散度定理及(2)式,有

$$\begin{aligned}\int_\Omega u^{\sigma-1}\Delta u^m\mathrm{d}x=&\int_{\partial\Omega}u^{\sigma-1}\frac{\partial u^m}{\partial n}\mathrm{d}A-\int_\Omega\nabla u^{\sigma-1}\cdot\nabla u^m\mathrm{d}x\leq mb_1\int_{\partial\Omega}u^{\sigma+m+s_1-2}\mathrm{d}A-\\&m(\sigma-1)\int_\Omega u^{\sigma+m-3}|\nabla u|^2\mathrm{d}x\leq\frac{mb_1n}{\rho_0}\int_\Omega u^{\sigma+m+s_1-2}\mathrm{d}x+\frac{(\sigma+m+s_1-2)dmb_1}{\rho_0}\cdot\\&\int_\Omega u^{\sigma+m+s_1-3}|\nabla u|\mathrm{d}x-\frac{4m(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega|\nabla u^{\frac{\sigma+m-1}{2}}|^2\mathrm{d}x.\end{aligned} \tag{33}$$

对于(33)式右边第 2 项,由 Hölder 不等式和 Young 不等式,得:

$$\int_\Omega u^{\sigma+m+s_1-3}\,|\nabla u|\,\mathrm{d}x \le \frac{1}{2\varepsilon_1}\int_\Omega u^{\sigma+m+2s_1-3}\mathrm{d}x + \frac{2\varepsilon_1}{(\sigma+m-1)^2}\int_\Omega |\nabla u^{\frac{\sigma+m-1}{2}}|^2\mathrm{d}x, \tag{34}$$

其中 ε_1 是正数，定义见后面文字.

于是，由(33)式和(34)式，得到：

$$\sigma k_1^\delta(t)\int_\Omega u^{\sigma-1}\Delta u^m\mathrm{d}x \le r_1k_1^\delta(t)\int_\Omega u^{\sigma+m+s_1-2}\mathrm{d}x + r_2k_1^\delta(t)\int_\Omega u^{\sigma+m+2s_1-3}\mathrm{d}x + r_3k_1^\delta(t)\int_\Omega |\nabla u^{\frac{\sigma+m-1}{2}}|^2\mathrm{d}x, \tag{35}$$

其中：$r_1=\frac{mb_1n}{\rho_0}\sigma$，$r_2=\frac{(\sigma+m+s_1-2)dmb_1}{\rho_0}\frac{\sigma}{2\varepsilon_1}$，

$r_3=\frac{(\sigma+m+s_1-2)dmb_1}{\rho_0}\frac{2\varepsilon_1\sigma}{(\sigma+m-1)^2}-\frac{4m(\sigma-1)\sigma}{(\sigma+m-1)^2}$.

对于(32)式右边第 5 项，由散度定理及(2)式，有

$$\int_\Omega v^{\sigma-1}(\rho(|\nabla v|^2)v_{,i})_{,i}\mathrm{d}x = \int_{\partial\Omega} v^{\sigma-1}\rho(|\nabla v|^2)\frac{\partial v}{\partial n}\mathrm{d}A - \int_\Omega \rho(|\nabla v|^2)\nabla v\cdot\nabla v^{\sigma-1}\mathrm{d}x \le \frac{nb_2}{\rho_0}\int_\Omega v^{\sigma+s_2-1}\mathrm{d}x + \frac{(\sigma+s_2-1)db_2}{\rho_0}\int_\Omega v^{\sigma+s_2-2}\,|\nabla v|\,\mathrm{d}x - \frac{4(\sigma-1)}{\sigma^2}\int_\Omega |\nabla v^{\frac{\sigma}{2}}|^2\mathrm{d}x. \tag{36}$$

对于(36)式右边第 2 项，由 Hölder 不等式和 Young 不等式，得：

$$\int_\Omega v^{\sigma+s_2-2}\,|\nabla v|\,\mathrm{d}x \le \frac{1}{2\varepsilon_2}\int_\Omega v^{\sigma+2s_2-2}\mathrm{d}x + \frac{2\varepsilon_2}{\sigma^2}\int_\Omega |\nabla v^{\frac{\sigma}{2}}|^2\mathrm{d}x, \tag{37}$$

其中 ε_2 是正数，定义见后面文字.

于是由(36)、(37)式有

$$\sigma k_2^\delta(t)\int_\Omega v^{\sigma-1}(\rho(|\nabla v|^2)v_{,i})_{,i}\mathrm{d}x \le \lambda_1k_2^\delta(t)\int_\Omega v^{\sigma+s_2-1}\mathrm{d}x + \lambda_2k_2^\delta(t)\int_\Omega v^{\sigma+2s_2-2}\mathrm{d}x + \lambda_3k_2^\delta(t)\int_\Omega |\nabla v^{\frac{\sigma}{2}}|^2\mathrm{d}x, \tag{38}$$

其中：$\lambda_1=\frac{nb_2\sigma}{\rho_0}$，$\lambda_2=\frac{(\sigma+s_2-1)db_2}{\rho_0}\frac{\sigma}{2\varepsilon_2}$，$\lambda_3=\frac{(\sigma+s_2-1)db_2}{\rho_0}\frac{2\varepsilon_2}{\sigma}-\frac{4(\sigma-1)}{\sigma}$.

对于(32)式右边第 3 项，由 Hölder 不等式和 Young 不等式，有

$$\sigma k_1^{\delta+1}(t)\int_\Omega u^{\sigma+p-1}\mathrm{d}x\int_\Omega v^q\mathrm{d}x \le \sigma\,|\Omega|\,k_1^{\delta+1}(t)\left(\int_\Omega u^{\sigma+p+q-1}\mathrm{d}x\right)^{\frac{\sigma+p-q}{\sigma+p+q-1}}\left(\int_\Omega v^{\sigma+p+q-1}\mathrm{d}x\right)^{\frac{q}{\sigma+p+q-1}} \le \tilde{k}(t)k_1^\delta(t)\int_\Omega u^{\sigma+p+q-1}\mathrm{d}x + \tilde{k}(t)k_2^\delta(t)\int_\Omega v^{\sigma+p+q-1}\mathrm{d}x, \tag{39}$$

其中 $\tilde{k}(t)=\sigma\,|\Omega|\,k_1^{\delta+1-\frac{\sigma+p-1}{\sigma+p+q-1}\delta}(t)k_2^{-\frac{q}{\sigma+p+q-1}\delta}(t)$.

类似地，对于(32)式右边第 6 项，由 Hölder 不等式和 Young 不等式，有

$$\sigma k_2^{\delta+1}(t)\int_\Omega v^{\sigma+r-1}\mathrm{d}x\int_\Omega u^s\mathrm{d}x \le \tilde{\tilde{k}}(t)k_2^\delta(t)\int_\Omega v^{\sigma+r+s-1}\mathrm{d}x + \tilde{\tilde{k}}(t)k_1^\delta(t)\int_\Omega u^{\sigma+r+s-1}\mathrm{d}x, \tag{40}$$

其中 $\tilde{\tilde{k}}(t)=\sigma\,|\Omega|\,k_2^{\delta+1-\frac{r+s-1}{\sigma+r+s-1}\delta}(t)k_1^{-\frac{s}{\sigma+r+s-1}\delta}(t)$.

联立(32)式、(35)式以及(38)～(40)式，得：

$$\begin{aligned}\phi'(t) \le{}& \delta a\phi(t) + r_1k_1^\delta(t)\int_\Omega u^{\sigma+m+s_1-2}\mathrm{d}x + r_2k_1^\delta(t)\int_\Omega u^{\sigma+m+2s_1-3}\mathrm{d}x - \sigma k_1^\delta(t)\int_\Omega u^{\sigma+\alpha-1}\mathrm{d}x +\\ &\tilde{k}(t)k_1^\delta(t)\int_\Omega u^{\sigma+p+q-1}\mathrm{d}x + \tilde{\tilde{k}}(t)k_1^\delta(t)\int_\Omega u^{\sigma+r+s-1}\mathrm{d}x + r_3k_1^\delta(t)\int_\Omega |\nabla u^{\frac{\sigma+m-1}{2}}|^2\mathrm{d}x +\\ &\lambda_1k_2^\delta(t)\int_\Omega v^{\sigma+s_2-1}\mathrm{d}x + \lambda_2k_2^\delta(t)\int_\Omega v^{\sigma+2s_2-2}\mathrm{d}x + \lambda_3k_2^\delta(t)\int_\Omega |\nabla v^{\frac{\sigma}{2}}|^2\mathrm{d}x +\\ &\tilde{k}(t)k_2^\delta(t)\int_\Omega v^{\sigma+p+q-1}\mathrm{d}x + \tilde{\tilde{k}}(t)k_2^\delta(t)\int_\Omega v^{\sigma+r+s-1}\mathrm{d}x - \sigma k_2^\delta(t)\int_\Omega v^{\sigma+\beta-1}\mathrm{d}x.\end{aligned} \tag{41}$$

由 Hölder 不等式和 Young 不等式,可得:

$$\int_\Omega u^{\sigma+m+s_1-2}\mathrm{d}x \le x_{10}\varepsilon_3\int_\Omega u^{\sigma+m+2s_1-3}\mathrm{d}x + x_{20}\varepsilon_3^{-\frac{x_{10}}{x_{20}}}\int_\Omega u^\sigma\mathrm{d}x, \tag{42}$$

$$\int_\Omega v^{\sigma+s_2-1}\mathrm{d}x \le y_{10}\varepsilon_4\int_\Omega v^{\sigma+2s_2-2}\mathrm{d}x + y_{20}\varepsilon_4^{-\frac{y_{10}}{y_{20}}}\int_\Omega v^\sigma\mathrm{d}x, \tag{43}$$

$$\int_\Omega u^{\sigma+m+2s_1-3}\mathrm{d}x \le x_{11}\varepsilon_5\int_\Omega u^{\sigma+\alpha-1}\mathrm{d}x + x_{21}\varepsilon_5^{-\frac{x_{11}}{x_{21}}}\int_\Omega u^\sigma\mathrm{d}x, \tag{44}$$

$$\int_\Omega v^{\sigma+2s_2-2}\mathrm{d}x \le y_{11}\varepsilon_6\int_\Omega v^{\sigma+\beta-1}\mathrm{d}x + y_{21}\varepsilon_6^{-\frac{y_{11}}{y_{21}}}\int_\Omega v^\sigma\mathrm{d}x, \tag{45}$$

其中:$x_{10}=\dfrac{m+s_1-2}{m+2s_1-3}$,$x_{20}=\dfrac{s_1-1}{m+2s_1-3}$,$y_{10}=\dfrac{1}{2}$,$y_{20}=\dfrac{1}{2}$,$x_{11}=\dfrac{m+2s_1-3}{\alpha-1}$,$x_{21}=\dfrac{\alpha-m-2s_1+2}{\alpha-1}$,$y_{11}=\dfrac{2s_2-2}{\beta-1}$,$y_{21}=\dfrac{\beta-2s_2+1}{\beta-1}$,$\varepsilon_3$,$\varepsilon_4$,$\varepsilon_5$,$\varepsilon_6$ 均为正数,定义见后面文字.

联立(41)~(45)式,可推出

$$\begin{aligned}\phi'(t) \le{}& (\delta a+K_1+K_2)\phi(t)+\tilde{k}(t)k_1^\delta(t)\int_\Omega u^{\sigma+p+q-1}\mathrm{d}x+\tilde{\tilde{k}}(t)k_1^\delta(t)\int_\Omega u^{\sigma+r+s-1}\mathrm{d}x+\\ & [x_{11}\varepsilon_5(x_{10}\varepsilon_3r_1+r_2)-\sigma]k_1^\delta(t)\int_\Omega u^{\sigma+\alpha-1}\mathrm{d}x+r_3k_1^\delta(t)\int_\Omega|\nabla u^{\frac{\sigma+m-1}{2}}|^2\mathrm{d}x+\\ & \lambda_3k_2^\delta(t)\int_\Omega|\nabla v^{\frac{\sigma}{2}}|^2\mathrm{d}x+\tilde{k}(t)k_2^\delta(t)\int_\Omega v^{\sigma+p+q-1}\mathrm{d}x+\tilde{\tilde{k}}(t)k_2^\delta(t)\int_\Omega v^{\sigma+r+s-1}\mathrm{d}x+\\ & [y_{11}\varepsilon_6(y_{10}\varepsilon_4\lambda_1+\lambda_2)-\sigma]k_2^\delta(t)\int_\Omega v^{\sigma+\beta-1}\mathrm{d}x,\end{aligned} \tag{46}$$

其中:$K_1=x_{20}\varepsilon_3^{-\frac{x_{10}}{x_{20}}}r_1+y_{20}\varepsilon_4^{-\frac{y_{10}}{y_{20}}}\lambda_1$,$K_2=x_{21}\varepsilon_5^{-\frac{x_{11}}{x_{21}}}(x_{10}\varepsilon_3r_1+r_2)+y_{21}\varepsilon_6^{-\frac{y_{11}}{y_{21}}}(y_{10}\varepsilon_4\lambda_1+\lambda_2)$.取合适的 ε_3,ε_4,ε_5,ε_6,使得 $x_{11}\varepsilon_5(x_{10}\varepsilon_3r_1+r_2)-\sigma\le 0$,$y_{11}\varepsilon_6(y_{10}\varepsilon_4\lambda_1+\lambda_2)-\sigma<0$. 于是,(46)式化为:

$$\begin{aligned}\phi'(t) \le{}& (\delta a+K_1+K_2)\phi(t)+\tilde{k}(t)k_1^\delta(t)\int_\Omega u^{\sigma+p+q-1}\mathrm{d}x+\tilde{\tilde{k}}(t)k_1^\delta(t)\int_\Omega u^{\sigma+r+s-1}\mathrm{d}x+\\ & r_3k_1^\delta(t)\int_\Omega|\nabla u^{\frac{\sigma+m-1}{2}}|^2\mathrm{d}x+\tilde{k}(t)k_2^\delta(t)\int_\Omega v^{\sigma+p+q-1}\mathrm{d}x+\\ & \tilde{\tilde{k}}(t)k_2^\delta(t)\int_\Omega v^{\sigma+r+s-1}\mathrm{d}x+\lambda_3k_2^\delta(t)\int_\Omega|\nabla v^{\frac{\sigma}{2}}|^2\mathrm{d}x.\end{aligned} \tag{47}$$

由 Hölder 不等式和(3)式,有

$$\begin{aligned}&\int_\Omega u^{\sigma+r+s-1}\mathrm{d}x \le \left(\int_\Omega u^{\frac{(\sigma+m-1)n}{n-2}}\mathrm{d}x\right)^{x_{12}}\left(\int_\Omega u^\sigma\mathrm{d}x\right)^{x_{22}} \le r_4\left(\int_\Omega u^{\sigma+m-1}\mathrm{d}x\right)^{\frac{nx_{12}}{n-2}}\left(\int_\Omega u^\sigma\mathrm{d}x\right)^{x_{22}}+\\ & r_4\left(\int_\Omega|\nabla u^{\frac{\sigma+m-1}{2}}|^2\mathrm{d}x\right)^{\frac{nx_{12}}{n-2}}\left(\int_\Omega u^\sigma\mathrm{d}x\right)^{x_{22}} \le r_5\int_\Omega u^{\sigma+m-1}\mathrm{d}x+r_7\int_\Omega|\nabla u^{\frac{\sigma+m-1}{2}}|^2\mathrm{d}x+\\ & (r_6+r_8)\left(\int_\Omega u^\sigma\mathrm{d}x\right)^{\frac{(n-2)x_{22}}{n-2-nx_{12}}} \le r_5\frac{m-1}{r+s-1}\int_\Omega u^{\sigma+r+s-1}\mathrm{d}x+\frac{1}{2}\tilde{r}_5\int_\Omega u^\sigma\mathrm{d}x+\\ & r_7\int_\Omega|\nabla u^{\frac{\sigma+m-1}{2}}|^2\mathrm{d}x+(r_6+r_8)\left(\int_\Omega u^\sigma\mathrm{d}x\right)^{\frac{(n-2)x_{22}}{n-2-nx_{12}}},\end{aligned} \tag{48}$$

其中:$x_{12}=\dfrac{(r+s-1)(n-2)}{2\sigma+(m-1)n}$,$x_{22}=\dfrac{2\sigma+(m-1)n-(r+s-1)(n-2)}{2\sigma+(m-1)n}$,$r_4=(C^{\frac{2n}{n-2}}2^{\frac{n}{n-2}-1})^{x_{12}}$,$r_5=r_4\dfrac{nx_{12}}{n-2}\varepsilon_7$,$\tilde{r}_5=2r_5\dfrac{r+s-m}{r+s-1}$,$r_6=r_4\dfrac{n-2-nx_{12}}{n-2}\varepsilon_7^{-\frac{nx_{12}}{n-2-nx_{12}}}$,$r_7=r_4\dfrac{nx_{12}}{n-2}\varepsilon_8$,$r_8=r_4\dfrac{n-2-nx_{12}}{n-2}\varepsilon_8^{-\frac{nx_{12}}{n-2-nx_{12}}}$,$\varepsilon_7$,$\varepsilon_8$ 均为正数,定义见后面文字.取 ε_7,使得 $r_5\dfrac{m-1}{r+s-1}=\dfrac{1}{2}$,于是(48)式化为:

$$\int_\Omega u^{\sigma+r+s-1}\mathrm{d}x \le \tilde{r}_5\int_\Omega u^\sigma\mathrm{d}x+2r_7\int_\Omega|\nabla u^{\frac{\sigma+m-1}{2}}|^2\mathrm{d}x+2(r_6+r_8)\left(\int_\Omega u^\sigma\mathrm{d}x\right)^{\frac{(n-2)x_{22}}{n-2-nx_{12}}}. \tag{49}$$

同样，可得：

$$\int_\Omega u^{\sigma+p+q-1}\mathrm{d}x \le \tilde r_{10}\int_\Omega u^\sigma \mathrm{d}x + 2r_{12}\int_\Omega |\nabla u^{\frac{\sigma+m-1}{2}}|^2\mathrm{d}x + 2(r_{11}+r_{13})\left(\int_\Omega u^\sigma\mathrm{d}x\right)^{\frac{(n-2)x_{23}}{n-2-nx_{13}}}, \tag{50}$$

其中：$x_{13}=\dfrac{(p+q-1)(n-2)}{2\sigma+(m-1)n}$，$x_{23}=\dfrac{2\sigma+(m-1)n-(p+q-1)(n-2)}{2\sigma+(m-1)n}$，$r_9=(C^{\frac{2n}{n-2}}2^{\frac{n}{n-2}-1})^{x_{13}}$，

$r_{10}=r_9\dfrac{nx_{13}}{n-2}\varepsilon_9$，$\tilde r_{10}=2r_{10}\dfrac{p+q-m}{p+q-1}$，$r_{11}=r_9\dfrac{n-2-nx_{13}}{n-2}\varepsilon_9^{-\frac{nx_{13}}{n-2-nx_{13}}}$，$r_{12}=r_9\dfrac{nx_{13}}{n-2}\varepsilon_{10}$，

$r_{13}=r_9\dfrac{n-2-nx_{13}}{n-2}\varepsilon_{10}^{-\frac{nx_{13}}{n-2-nx_{13}}}$，$\varepsilon_9=\dfrac{(n-2)(p+q-1)}{2r_9nx_{13}(m-1)}$，$\varepsilon_{10}$ 为正数，定义见后面文字.

类似于(48)式的推导，由 Hölder 不等式和(4)式可得：

$$\int_\Omega v^{\sigma+r+s-1}\mathrm{d}x \le \lambda_4\left(\int_\Omega v^\sigma\mathrm{d}x\right)^{\frac{ny_{12}}{n-2}+y_{22}} + \lambda_5\int_\Omega |\nabla v^{\frac{\sigma}{2}}|^2\mathrm{d}x + \lambda_6\left(\int_\Omega v^\sigma\mathrm{d}x\right)^{\frac{(n-2)y_{22}}{n-2-ny_{12}}}, \tag{51}$$

$$\int_\Omega v^{\sigma+p+q-1}\mathrm{d}x \le \lambda_7\left(\int_\Omega v^\sigma\mathrm{d}x\right)^{\frac{ny_{13}}{n-2}+y_{23}} + \lambda_8\int_\Omega |\nabla v^{\frac{\sigma}{2}}|^2\mathrm{d}x + \lambda_9\left(\int_\Omega v^\sigma\mathrm{d}x\right)^{\frac{(n-2)y_{23}}{n-2-ny_{13}}}, \tag{52}$$

其中：$y_{12}=\dfrac{(r+s-1)(n-2)}{2\sigma}$，$y_{22}=\dfrac{2\sigma-(r+s-1)(n-2)}{2\sigma}$，$\lambda_4=(C^{\frac{2n}{n-2}}2^{\frac{n}{n-2}-1})^{y_{12}}$，$\lambda_5=\lambda_4\dfrac{ny_{12}}{n-2}\varepsilon_{11}$，

$\lambda_6=\lambda_4\dfrac{n-2-ny_{12}}{n-2}\varepsilon_{11}^{-\frac{ny_{12}}{n-2-ny_{12}}}$，$y_{13}=\dfrac{(p+q-1)(n-2)}{2\sigma}$，$y_{23}=\dfrac{2\sigma-(p+q-1)(n-2)}{2\sigma}$，

$\lambda_7=(C^{\frac{2n}{n-2}}2^{\frac{n}{n-2}-1})^{y_{13}}$，$\lambda_8=\lambda_7\dfrac{ny_{13}}{n-2}\varepsilon_{12}$，$\lambda_9=\lambda_7\dfrac{n-2-ny_{13}}{n-2}\varepsilon_{12}^{-\frac{ny_{13}}{n-2-ny_{13}}}$，$\varepsilon_{11}$，$\varepsilon_{12}$ 均为正数，定义见后面文字.

联立(47)式和(49)～(52)式，得：

$$\begin{aligned}\phi'(t) \le{}& (\delta a+K_1+K_2+K_3)\phi(t)+K_4k_1^\delta(t)\int_\Omega|\nabla u^{\frac{\sigma+m-1}{2}}|^2\mathrm{d}x+K_5k_2^\delta(t)\int_\Omega|\nabla v^{\frac{\sigma}{2}}|^2\mathrm{d}x+\\ & 2(r_6+r_8)\tilde{\tilde k}(t)k_1^\delta(t)\left(\int_\Omega u^\sigma\mathrm{d}x\right)^{\frac{(n-2)x_{22}}{n-2-nx_{12}}}+2(r_{11}+r_{13})\tilde k(t)k_1^\delta(t)\left(\int_\Omega u^\sigma\mathrm{d}x\right)^{\frac{(n-2)x_{23}}{n-2-nx_{13}}}+\\ & \lambda_4\tilde{\tilde k}(t)k_2^\delta(t)\left(\int_\Omega v^\sigma\mathrm{d}x\right)^{\frac{ny_{12}}{n-2}+y_{22}}+\lambda_6\tilde{\tilde k}(t)k_2^\delta(t)\left(\int_\Omega v^\sigma\mathrm{d}x\right)^{\frac{(n-2)y_{22}}{n-2-ny_{12}}}+\\ & \lambda_7\tilde k(t)k_2^\delta(t)\left(\int_\Omega v^\sigma\mathrm{d}x\right)^{\frac{ny_{13}}{n-2}+y_{23}}+\lambda_9\tilde k(t)k_2^\delta(t)\left(\int_\Omega v^\sigma\mathrm{d}x\right)^{\frac{(n-2)y_{23}}{n-2-ny_{13}}},\end{aligned} \tag{53}$$

其中：$K_3=\tilde r_5\tilde{\tilde k}(t)+\tilde r_{10}\tilde k(t)$，$K_4=r_3+2r_{12}\tilde k(t)+2r_7\tilde{\tilde k}(t)$，$K_5=\lambda_3+\lambda_5\tilde{\tilde k}(t)+\lambda_8\tilde k(t)$.

选取合适的 ε_1，ε_2，ε_8，ε_{10}，ε_{11}，ε_{12}，使得 $K_4\le 0$，$K_5\le 0$. 于是，(53)式化为：

$$\phi'(t)\le K_6\phi(t)+K_7(t)\phi(t)^{\xi_1}+K_8(t)\phi(t)^{\xi_2}+K_9(t)\phi(t)^{\xi_3}, \tag{54}$$

其中：$K_6=\delta a+K_1+K_2+K_3$，$\xi_1=\max\left\{\dfrac{(n-2)x_{22}}{n-2-nx_{12}},\dfrac{(n-2)y_{22}}{n-2-ny_{12}}\right\}>1$，

$\xi_2=\max\left\{\dfrac{(n-2)x_{23}}{n-2-nx_{13}},\dfrac{(n-2)y_{23}}{n-2-ny_{13}}\right\}>1$，$\xi_3=\max\left\{\dfrac{ny_{12}}{n-2}+y_{22},\dfrac{ny_{13}}{n-2}+y_{23}\right\}>1$，

$K_7(t)=2(r_6+r_8)\tilde{\tilde k}(t)k_1^{-\frac{2x_{12}}{n-2-nx_{12}}\delta}(t)+\lambda_6\tilde{\tilde k}(t)k_2^{-\frac{2y_{12}}{n-2-ny_{12}}\delta}(t)$，

$K_8(t)=2(r_{11}+r_{13})\tilde{\tilde k}(t)k_1^{-\frac{2x_{13}}{n-2-nx_{13}}\delta}(t)+\lambda_9\tilde k(t)k_2^{-\frac{2y_{13}}{n-2-ny_{13}}\delta}(t)$，

$K_9(t)=\lambda_4\tilde{\tilde k}(t)k_1^{-\frac{2y_{12}}{n-2}\delta}(t)+\lambda_7\tilde k(t)k_2^{-\frac{2y_{13}}{n-2}\delta}(t)$，设 $\Theta(t^*)=\int_0^{t^*}K(\tau)\mathrm{d}\tau$，其中 $K(t)=1+K_7(t)+K_8(t)+K_9(t)$.

对(54)式从 0 到 t^* 积分，有

$$\Theta(t^*) \geq \int_{\phi(0)}^{\infty} \frac{1}{K_6 \eta + \eta^{\xi_1} + \eta^{\xi_2} + \eta^{\xi_3}} \mathrm{d}\eta = S. \tag{55}$$

因为 $\xi_i > 1(i=1,2,3)$,所以(55)式右边积分存在.易知,$\Theta(t^*)$是单调递增函数,于是有

$$t^* \geq \Theta^{-1}(S), \tag{56}$$

其中 Θ^{-1} 是 Θ 的反函数,于是完成了定理 2 的证明.

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Blow-up of solutions to a nonlocal mixed parabolic system with time-dependent coefficients

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Abstract: Blow-up of solutions to a nonlocal mixed parabolic system with time-dependent coefficients and inner absorption terms under nonlinear boundary conditions is studied. By using a differential technique, the sufficient conditions for the global existence for a nonlocal mixed parabolic system with time-dependent coefficients and inner absorption terms under nonlinear boundary conditions in high spaces are obtained. Furthermore, the lower bound estimate of blow-up time is derived by formulating energy expressions and using Sobolev inequalities and other differential methods.

Keywords: blow-up; parabolic system; global existence; time-dependent coefficient; absorption term

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